# FAULT DETECTION AND ESTIMATION IN A CLASS OF DISTRIBUTED PARAMETER SYSTEMS

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**Abstract**: In this paper, the problem of fault detection and isolation in a class of distributed parameter systems (DPS) will be investigated. The behavior of distributed parameter systems is best described by partial differential equation (PDE) models. However, due to complex nature of DPS, a PDE model is traditionally transformed into a finite set of ordinary differential equations (ODE) prior to the design of control or fault detection schemes by using significant approximations thus reducing the accuracy and reliability of the overall system. By contrast, in this paper, the PDE representation of the system is directly utilized to design the fault diagnosis scheme for DPS. Faults that can occur anywhere in the domain of the DPS (referred to as state faults) are considered, rather than only actuator and sensor faults. State faults are significantly more complicated to deal with in the case of DPS since they can be initiated anywhere within a continuous range of space, while in practice sensors are only available at limited locations which in many cases only include the input and/or output sides of the DPS. This problem is tackled by using an observer structure, which includes input, and output filters directly based on the PDE model of the system. A fault is detected by comparing the detection residual, which is the difference between measured and estimated outputs, with a predefined detection threshold. Once the fault is detected, an online approximator is activated to learn the fault function. An update law is introduced for updating the unknown parameters of the online approximator. The stability of the observer along with the online approximator will be discussed analytically in the paper. It is shown that one sensor is satisfactory for fault detection and approximation if the fault function has only one unknown parameter or can be expressed as linear in the unknown parameters. However, additional sensors are required for fault approximation or isolation in the general case. For example, a leakage fault in a pipeline has magnitude and location as unknown parameters and these parameters cannot be approximated by using one sensor. An algorithm is designed to approximate the location of a state fault with unknown magnitude by using multiple sensors. The distributed parameter systems considered in this paper are modelled by parabolic partial differential equations with Neumann or Dirichlet boundary conditions. Heat transfer systems and fluid pipelines are examples of systems in this class of DPS. The scheme is verified in simulations on the aforementioned systems.

**Key words**: Diagnostics; prognostics; distributed parameter systems

**Introduction**: Fault diagnosis has become an attractive research topic in the past couple of decades due to increased complexity of industrial systems and safety for such systems has become more important than ever. Among the different methods of fault diagnosis, model-based methods have become both popular and more suitable when a mathematical model of the system under consideration is either available or can be obtained [1] because they do not need extensive amount of offline data and can operate online without requiring additional sensors. Therefore, model-based fault diagnosis schemes have been developed for lumped parameter systems based on their ordinary differential equation (ODE) representation by using sliding mode observers [2, 3], online approximators [4], geometric approach [5], and adaptive observers [6, 7].

A large number of industrial systems which involve heat transfer, fluid dynamics, and others are classified as distributed parameter systems (DPS). Application examples of such DPS include hydraulic systems, chemical processes, flexible robots, and aerospace systems. Due to the wide range of such systems and their important and sensitive role in the industry, reliable fault detection and diagnosis schemes are required to guarantee their safe operation.

The variables in DPS are defined over a continuous range of space [8], which makes them different from lumped parameter systems where each variable only evolves in time. The most comprehensive and accurate mathematical representation for these systems is given in terms of partial differential equations (PDEs). Limited work has been done on DPS when compared to the systems with ODE models, because dealing with PDEs is much more complicated due to boundary conditions and infinite number of states [8].

In order to simplify the problem of dealing with PDE model, an infinite set of ordinary differential equations (ODEs) is employed and then the Galerkin's method is applied to obtain an approximate finite dimensional ODE model [9, 10] which is subsequently utilized for fault detection and diagnosis. A learning systems approach is introduced in [11] for fault detection of such systems whereas fault detection and isolation of such DPS with actuator faults are discussed in [12], while a geometric approach is proposed for fault detection and isolation of dissipative parabolic PDEs [13].

Although this method has opened new doors to the problem of control and fault detection of DPS, it has a number of challenges. First of all, these methods [11-13] can possibly render inaccurate results since they utilize an approximate system dynamics by neglecting a significant portion of the system dynamics. In addition, there is no guarantee that the system output can be generated as a function of only the states of the finite dimensional ODE system. Further, when a fault happens in the system, the PDE dynamics will change, which can make the approximated ODE model even more inaccurate.

In this paper, a class of DPS modeled by linear parabolic PDE is investigated. Faults that are considered here are state faults which can occur anywhere in the domain of the DPS and affect the PDE dynamics of the system rather than faults which only affect input or outputs through boundary conditions. State faults are significantly more complicated to deal with in the case of DPS since they can be initiated anywhere within a continuous range of space, while in practice sensors are only available at limited locations which in many cases only include the input and/or output sides of the DPS. This problem is tackled by using an observer structure which includes input, output, and fault filters. The combination of input and output filters provides an estimate of the system behavior under healthy conditions while the fault filters are added upon detection to help in fault approximation based on the assumption that the fault can be represented as a linear combination of known basis functions. Therefore, in contrast with existing ODE-based schemes [11-13], the detection observer is designed directly based on the PDE model and the aforementioned filters. It is shown that the proposed observer can estimate both measured and unmeasured system states in the healthy operating conditions with a bounded error. Detection residual is generated by comparing the measured and estimated system outputs. Since the residual is bounded in healthy conditions, a fault can be detected by comparing it with a predefined detection threshold. It should be noted that in earlier works of the author fault diagnosis and fault accommodation schemes have been developed for DPS directly based on the PDE model of the system [14,15]. However, the schemes presented in [14,15] require the distributed state to be measureable at all points, which is a very strong assumption for DPS. In contrast, the scheme presented in this paper only requires the boundary values of the system state, providing a more practical method for fault diagnosis in DPS.

Upon detecting a fault, a number of fault filters, which reflect the effect of fault basis functions on the system behavior are activated and incorporated in the PDE observer with adaptive weights. The particular fault that has occurred will be approximated as a weighted summation of fault basis functions, where the adaptive weights are tuned by an adaptive update law, which seeks to minimize the error between observer output and measured output. The stability of both the proposed observer and fault approximator is investigated analytically.

The effectiveness and stability of the proposed scheme is verified by using a heat system and a fluid pipeline. The dynamics of both systems are described by parabolic PDEs. The proposed PDE observer is utilized to estimate the system states and provide an estimation of the system output which is used to generate detection residual. Fault detection and estimation are successfully performed in simulations and the results are presented in the last section of this paper.

The paper is organized as follows: First the system and its model are described mathematically. Then the PDE observer and online estimation of fault dynamics are presented, and finally the verification of proposed scheme in simulations is provided.

**System Description**: Consider the class of linear system described by the following normalized PDE

$$\frac{\partial \overline{x}\left(\overline{z},t\right)}{\partial t}=\overline{a}\,\frac{\partial^2 \overline{x}\left(\overline{z},t\right)}{\partial \overline{z}^2}+\overline{b}\,\frac{\partial \overline{x}\left(\overline{z},t\right)}{\partial \overline{z}}+\overline{c}\,\overline{x}\left(\overline{z},t\right), \tag{1}$$

for $(\bar{z}, \bar{t}) \in (0, \bar{L}) \times \mathbb{R}^{+}$, with the following boundary conditions

$$\begin{cases} \dfrac{\partial \bar{x}(0,t)}{\partial \bar{z}} = \bar{q}\,\bar{x}(0,t), \\ \dfrac{\partial \bar{x}(L,t)}{\partial \bar{z}} = \bar{u}(t) \qquad or \qquad \bar{x}(L,t) = \bar{u}(t) \end{cases} \tag{2}$$

where $\bar{x} \in (0, L) \times \mathbb{R}$ is the state of the system, $\bar{a}, \bar{b}, \bar{c}$, and $\bar{q}$ are scalar constants, and $\bar{u} \in \mathbb{R}$ is the control input applied as either Dirichlet or Neumann boundary control at $\bar{z} = L$. Suppose that the output of the system is at the opposite end of the actuation, i.e. the output $\bar{y} \in \mathbb{R}$ which is the only available measurement given by

$$\bar{y}(t) = \bar{x}(0,t)\,. \tag{3}$$

Now the PDE in (1) can be normalized and $\bar{b}$ can be eliminated from the equation by using the following transformation [16]

$$\bar{x}(\bar{z},t) \to x(z,t)e^{-\frac{bz}{2a}}.$$

After applying this transformation, the system will be described by

$$\frac{\partial x(z,t)}{\partial t} = a\frac{\partial^2 x(z,t)}{\partial z^2} + cx(z,t), \tag{4}$$

$$\begin{cases} \dfrac{\partial x(0,t)}{\partial z} = qx(0,t), \\ \dfrac{\partial x(1,t)}{\partial z} = u(t) \qquad or \qquad x(1,t) = u(t), \end{cases} \tag{5}$$

$$y(t) = x(0,t)\,, \tag{6}$$

for $(z,t) \in (0,1) \times \mathbb{R}^+$, where $x \in (0,1) \times \mathbb{R}$ is the system state, and $a$, $c$, and $q$ are scalar constants such that $c < a\pi^2/4$. Next consider a fault in the domain of the DPS which is modeled by $h(z,t)$. Then the system representation in (4) and (5) can be rewritten in the presence of a fault as

$$\frac{\partial x(z,t)}{\partial t} = a\frac{\partial^2 x(z,t)}{\partial z^2} + cx(z,t) + h(z,t), \tag{7}$$

subject to the boundary conditions

$$\begin{cases} \dfrac{\partial x(0,t)}{\partial z} = qx(0,t), \\ \dfrac{\partial x(1,t)}{\partial z} = u(t) \qquad or \qquad x(1,t) = u(t). \end{cases} \tag{8}$$

Moreover, the fault function can be represented by

$$h(z,t) = \Omega(t-t_0)\bar{h}(z,t),$$

where $\Omega(t - t_0)$ is the time profile of the fault defined by

$$\Omega(^2) = \begin{cases} 0 & , \; if\,^2 < \\ 1 - e^{-©^2} & , \; if\,^2 \geq \end{cases} ,$$

with $\kappa \in \mathbb{R}^+$ is an unknown constant that is determined by the growth rate of the fault. Although, this time profile is basically used to model incipient faults, it can also address abrupt faults for large values of $\kappa$. The following standard assumption is needed in order to proceed.

Assumption 1: The fault function can be expressed as linear in the unknown parameters (LIP) [17], i.e. $h = w^T o(y, z)$ where $w \in \mathbb{R}^p$ is the unknown parameters vector and $o(\cdot)$ is a known nonlinear vector function which is bounded by $\|o\| \le o_m$ .

**Fault Detection Observer**: In order to detect a fault, an observer is utilized to estimate the system output in healthy conditions. Then the estimated and measured outputs will be compared to generate fault detection residual. For the purpose of observer development based on the PDE model of the system in healthy operating conditions, we first investigate the healthy system dynamics given in (4) and (5). Since (4) and (5) define a linear PDE system, its solution is the summation of system response due to initial conditions plus the system response due to external inputs. If we consider both the boundary terms $q(0, t)$ and $u(t)$ as external inputs, then the solution $x(z, t)$ can be obtained as

$$x(z,t) = x_0(z,t) + x_1(z,t) + x_2(z,t), \tag{9}$$

where $x_0(z, t)$ is the response of the system due to initial conditions with zero external inputs, i.e. the solution of

$$\frac{\partial x_0(z,t)}{\partial t} = a\frac{\partial^2 x_0(z,t)}{\partial z^2} + c x_0(z,t), \tag{10}$$

$$\begin{cases} \dfrac{\partial x_0(0,t)}{\partial z} = 0, \\ \dfrac{\partial x_0(1,t)}{\partial z} = 0 \qquad or \qquad x_0(1,t) = 0, \end{cases} \tag{11}$$

$$x_0(z,0) = x(z,0), \tag{12}$$

with $x_1(z, t)$ is the response of the system due to $q(0, t) = q(t)$, i.e. the solution of

$$\frac{\partial x_1(z,t)}{\partial t} = a\frac{\partial^2 x_1(z,t)}{\partial z^2} + c x_1(z,t), \tag{13}$$

$$\begin{cases} \dfrac{\partial x_1(0,t)}{\partial z} = qy(t), \\ \dfrac{\partial x_1(1,t)}{\partial z} = 0 \qquad or \qquad x_1(1,t) = 0, \end{cases} \tag{14}$$

$$x_1(z,0) = 0, \tag{15}$$

and $x_2(z, t)$ is the response of the system due to $u(t)$, i.e. the solution of

$$\frac{\partial x_2(z,t)}{\partial t} = a\frac{\partial^2 x_2(z,t)}{\partial z^2} + c x_2(z,t), \tag{16}$$

$$\begin{cases} \dfrac{\partial x_2(0,t)}{\partial z} = 0, \\ \dfrac{\partial x_2(1,t)}{\partial z} = u(t) \qquad or \qquad x_2(1,t) = u(t), \end{cases} \tag{17}$$

$$x_2(z,0) = 0. \tag{18}$$

The system defined by (13-15) is called an output filter [16] since it is derived by the output of the actual system, while the system defined by (16-18) is called an input filter [16] since it is derived by the input of the actual system.

The fault detection observer is now designed based on this discussion. Since the system dynamics are known and the external inputs that are deriving the systems (13-15) and (16-18) are both available, $x_1(z,t)$ and $x_2(z,t)$ can be obtained for $0 \leq z \leq 1$ when $t > 0$. If the initial condition $x(z,0)$ was available for all values of $0 \leq z \leq 1$, then $x_c(z,t)$ could also be obtained, giving us an exact estimation of system state using equation (9). However, the initial conditions are generally not known, thus $x_c(z,t)$ cannot be used in the observer development. But if the system defined by (10) and (11) is asymptotically stable, then its solution $x_c(z,t)$ will converge to zero, so the state and output estimates provided by the following observer

$$\hat{x}(z,t) = x_1(z,t) + x_2(z,t), \tag{19}$$

$$\hat{y}(t) = \hat{x}(0,t), \tag{20}$$

where $\hat{x} \in (0,1) \times \mathbb{R}$ and $\hat{y} \in \mathbb{R}$ are the observer state and output respectively, will converge to the true value of the system state and output asymptotically.

The detection residual $e \in \mathbb{R}$ is defined as the difference between actual and estimated system outputs, i.e.

$$e(t) = y(t) - \hat{y}(t). \tag{21}$$

Fault detection is performed by comparing this detection residual with a predefined detection threshold which is derived. A fault is detected when the absolute value of the detection residual $|e|$ exceeds the detection threshold $\rho \in \mathbb{R}^+$. The next theorem discusses the boundedness of residual under healthy conditions and selection of detection threshold.

Theorem 1 (Fault Detection Observer Performance): Let the observer in (19) and (20) be used to monitor the system in (4)-(6). If the threshold for the detection residual is selected as

$$\circ = |e(0)|\left( e^{-\frac{a\text{®}^2 - 4c}{4}t} + \text{¦} \right), \tag{22}$$

then a fault will not be detected as long as the system is working under healthy operating conditions.

Proof: First define the state residual $\tilde{x} \in (0,1) \times \mathbb{R}$ as the difference between the actual and estimated state $\tilde{x} = x - \hat{x}$. When the system is working in the healthy condition, the state residual is given by $\tilde{x}(z,t) = x_0(z,t)$. Therefore, the state residual dynamics in the case of Neumann boundary are described by

$$\frac{\partial \tilde{x}(z,t)}{\partial t} = a\frac{\partial^2 \tilde{x}(z,t)}{\partial z^2} + c\tilde{x}(z,t),$$

$$\begin{cases} \dfrac{\partial \tilde{x}(0,t)}{\partial z} = 0 \\ \dfrac{\partial \tilde{x}(1,t)}{\partial z} = 0 \end{cases},$$

with initial condition $\tilde{x}(z,0) = x(z,0)$. To show the stability of residual dynamics during the healthy conditions, select the following positive definite and radially unbounded Lyapunov function candidate

$$V(t) = \frac{1}{2}\|\tilde{x}\|^2 = \frac{1}{2}\int_0^1 \tilde{x}^2(z,t)\,dz .$$

The derivative of this Lyapunov function candidate is given by

$$\dot{V}(t) = \int_0^1 \frac{\partial \tilde{x}(z,t)}{\partial t}\tilde{x}(z,t)\,dz = \int_0^1 \left( a\frac{\partial^2 \tilde{x}(z,t)}{\partial z^2} + c\tilde{x}(z,t) \right)\tilde{x}(z,t)\,dz.$$

By applying integration by parts and then Poincare inequality [18], it can be shown that

$$\dot{V}(t) = -a\left\|\frac{\partial \tilde{x}(z,t)}{\partial z}\right\|^2 + c\|\tilde{x}\|^2 \le -\frac{a\pi^2 - 4c}{4}\|\tilde{x}\|^2 .$$

Since $c < a\pi^2/4$, it can be concluded that the state residual is globally exponentially stable. Therefore, the residual always satisfies

$$|e(t)| \le |e(0)|\,e^{-\frac{a\pi^2-4c}{4}t}.$$

To avoid false alarms, a constant threshold can be selected as $\rho = (\eta + 1)|e(0)|$, or a time varying threshold can be selected as

$$\rho = |e(0)|\left( e^{-\frac{a\pi^2-4c}{4}t} + \eta \right),$$

where $\eta$ is a small positive constant which is used in the threshold to ensure that $|e(t)|$ remains below the threshold at all times during healthy operation.

Remark 1: Theorem 1 defines a detection threshold in the absence of modeling uncertainty or noise. In the presence of uncertainties and noise, their upper bounds must be included in the derivation of detection threshold in order to prevent false alarms. Therefore, dealing with modeling uncertainties is part of the future work.

The next step in fault diagnosis is to determine the behavior of fault or approximating its dynamics, which allows further analysis of fault as well as estimation of remaining useful life of the system. To this end, an online approximator is added to the observer, which is discussed in the next section.

**Online Fault Estimation**: As mentioned above, the proposed detection observer is able to estimate the distributed system states and output with an asymptotically decreasing error in healthy operating conditions. When a fault occurs, the residual is no longer bounded and the fault is detected when the residual exceeds detection threshold. Upon detecting a fault, adaptive filters are added to the detection observer in order to estimate the fault parameters.

In the presence of fault, the system dynamics are described by the following PDE

$$\frac{\partial x(z,t)}{\partial t} = a\frac{\partial^2 x(z,t)}{\partial z^2} + cx(z,t) + \sum_{i=1}^{p} w_i \pm_i (y,z), \tag{23}$$

for $(z,t) \in (0,1) \times \mathbb{R}^+$, where $x \in (0,1) \times \mathbb{R}$ is the system state, $o = [o_1 \ldots o_p]^T$ is the vector of fault basis functions, and $w = [w_1 \ldots w_p]^T$ is the vector of unknown fault parameters. Similar to the design of input and output filters in the previous section, we now consider the terms $w_i^T o_i(y,z)$ for $i = 1, \ldots, p$ as external outputs to the system. Now we construct *p* fault filters to determine the response of system to each fault basis function. The *i*th fault filter is designed as

$$\frac{\partial \text{¤}_i(z,t)}{\partial t} = a\frac{\partial^2 \text{¤}_i(z,t)}{\partial z^2} + c\text{¤}_i(z,t) + \pm_i(y,z), \tag{24}$$

$$\begin{cases} \dfrac{\partial \text{¤}_i(0,t)}{\partial z} = 0, \\ \dfrac{\partial \text{¤}_i(1,t)}{\partial z} = 0 \qquad or \qquad \text{¤}_i(1,t) = 0, \end{cases} \tag{25}$$

$$\text{¤}_i(z,0) = 0 \tag{26}$$

for $(z,t) \in (0,1) \times \mathbb{R}^+$ and $i = 1 \ldots p$, where $\phi_i \in (0,1) \times \mathbb{R}$ is the *i*th fault filter state. With this definition the system state in the presence of fault can be represented as

$$x(z,t) = x_0(z,t) + x_1(z,t) + x_2(z,t) + \sum_{i=1}^{p} w_i \text{¤}_i(z,t). \tag{27}$$

The fault basis functions are known, so the solution of fault filters can be obtained and used in the observer design. However, the fault parameter $w_i$ is not known, thus its estimated value $\hat{w}_i$ must be used in the diagnosis observer as follows

$$\hat{x}(z,t) = x_1(z,t) + x_2(z,t) + \sum_{i=1}^{p} \hat{w}_i(t)\text{¤}_i(z,t), \tag{28}$$

$$\hat{y}(t) = \hat{x}(0,t), \tag{29}$$

for $(z,t) \in (0,1) \times \mathbb{R}^+$ and $i = 1 \ldots p$. The adaptive update law for tuning the adaptive parameters to show boundedness of the residual and parameter estimation errors is provided in the next theorem.

Theorem 2 (Fault Diagnosis Observer Performance): Let the observer defined by (28) and (29) be utilized to monitor the DPS upon detecting a fault. If parameter update law is selected as

$$\dot{\hat{w}}(t) = \sim\Phi(t)e(t) - \text{¥}\,\hat{w}(t), \tag{30}$$

where $\Phi(t) = [\phi_1(0,t), \ldots, \phi_p(0,t)]^T$, $\hat{w} = [\hat{w}_1 \ldots \hat{w}_p]^T$, $\gamma$ is a positive constant, and $\alpha$ is the learning rate chosen such that

$$\sim < \frac{2a\text{®}^2 - 8c}{\text{®}^2 + 4}, \tag{31}$$

then the residual $e$ and the parameter estimation error $\tilde{w} = w - \hat{w}$ are uniformly ultimately bounded.

Proof: To prove the boundedness of estimation errors, a Lyapunov function candidate is chosen as

$$V(t) = \frac{1}{2}\|x_0\|^2 + \frac{1}{2}\sum_{i=1}^{p}\|\text{¤}_i\|^2 + \frac{1}{2}\tilde{w}^T\tilde{w}.$$

Derivative of this Lyapunov function with respect to time is given by

$$\begin{aligned}
\dot{V}(t) &= \int_0^1 \frac{\partial x_0(z,t)}{\partial t} x_0(z,t)\,dz - \tilde{w}^T(t)\dot{\hat{w}}(t) + \sum_{i=1}^{p}\int_0^1 \frac{\partial \text{¤}_i(z,t)}{\partial t}\text{¤}_i(z,t)\,dz \\
&= \int_0^1 \left( a\frac{\partial^2 x_0(z,t)}{\partial z^2} + c x_0(z,t) \right) x_0(z,t)\,dz - \tilde{w}^T(t)\dot{\hat{w}}(t) \\
&\quad + \sum_{i=1}^{p}\int_0^1 \left( a\frac{\partial^2 \text{¤}_i(z,t)}{\partial z^2} + c\text{¤}_i(z,t) + \pm_i(y,z) \right)\text{¤}_i(z,t)\,dz.
\end{aligned}$$

By applying integration by parts and Poincare inequality, and using the adaptive update law given in (30), it can be shown that

$$\begin{aligned}
\dot{V}(t) \le& -\frac{a\text{®}^2 - 4c}{4}\|x_0\|^2 - \frac{a\text{®}^2 - 4c}{4}\sum_{i=1}^{p}\left( \|\text{¤}_i\| - \frac{2\pm_{i_{\max}}}{a\text{®}^2 - 4c} \right)^2 \\
& - \sim\tilde{w}^T(t)\Phi(t)e(t) + \yen\,\tilde{w}^T(t)\hat{w}(t) + \frac{\pm_{\max}^2}{a\text{®}^2 - 4c}
\end{aligned}$$

where $\sigma_m^2 = \sum_{i=1}^{p}\sigma_{i_m}^2$. Since $e(t) = x(0,t) - \hat{x}(0,t) = x_0(0,t) + \tilde{w}^T(t)\Phi(\mathrm{t})$, the above inequality can be rewritten as

$$\begin{aligned}
\dot{V}(t) \le& -\frac{a\text{®}^2 - 4c}{4}\|x_0\|^2 - \frac{a\text{®}^2 - 4c}{4}\sum_{i=1}^{p}\left( \|\text{¤}_i\| - \frac{2\pm_{i_{\max}}}{a\text{®}^2 - 4c} \right)^2 \\
& - \sim\left(e(t) + x_0(0,t)\right)e(t) + \frac{\yen}{\sim}\tilde{w}^T(t)\left(w - \tilde{w}(t)\right) + \frac{\pm_{\max}^2}{a\text{®}^2 - 4c} \\
\le& -\left( \frac{a\text{®}^2 - 4c}{4} - \frac{\text{®}^2 + 4}{8}\sim \right)\|x_0\|^2 - \frac{\yen}{2\sim}\|\tilde{w}\|^2 \\
& - \frac{a\text{®}^2 - 4c}{4}\sum_{i=1}^{p}\left( \|\text{¤}_i\| - \frac{2\pm_{i_{\max}}}{a\text{®}^2 - 4c} \right)^2 + \frac{\yen}{2\sim}w_{\max}^2 + \frac{\pm_{\max}^2}{a\text{®}^2 - 4c}
\end{aligned}$$

With $\alpha$ being selected such that condition (31) holds, the derivative of Lyapunov function will be less than zero if the following conditions are satisfied:

$$\|x_0\| > \sqrt{\frac{8D}{2a\text{®}^2 - 8c - \text{\textasciitilde}\text{®}^2 - 4\text{\textasciitilde}}} \quad or$$

$$\|\tilde{w}\| > \sqrt{\frac{2\text{\textasciitilde}\, D}{\text{¥}}} \qquad or$$

$$\|\text{¤}_i\| > \sqrt{\frac{4D}{\text{\textasciitilde}\text{®}^2 - 4c}} + \frac{2\text{±}_{i_{\max}}}{a\text{®}^2 - 4c} \quad (i = 1, \ldots, p),$$

where $D = \frac{\gamma}{2a} w_m^2 \quad + \frac{\sigma_m^2}{a\pi^2 - 4c}$. Therefore, $x_c$, $\tilde{w}$ , and $\phi_i$ for $i = 1, \ldots, p$ are uniformly ultimately bounded, based on which the boundedness of $e(t) = x_c(0,t) + \tilde{w}^T(t)\Phi(t)$ is immediately followed.

**Simulation Results**: To verify the proposed fault detection and prognosis scheme, it has been applied in simulations on two different distributed parameter systems that are modeled by parabolic partial differential equations. First a heat system and then a fluid pipeline will be investigated.

Heat system: The normalized dynamics of a heat transfer system is given by

$$\frac{\partial x(z,t)}{\partial t} = \frac{\partial^2 x(z,t)}{\partial z^2} + 0.2x(z,t), \tag{33}$$

$$\begin{cases} \dfrac{\partial x(0,t)}{\partial z} = -0.4x(0,t), \\ x(1,t) = u(t), \end{cases} \tag{34}$$

$$y(t) = x(0,t), \tag{35}$$

for $z \in (0,1)$ and $t > 0$. The distributed system state $x(z,t)$ is the temperature. The input $u(t)$ in the form of a heater is applied at the position $z = 1$ while the only available measurement $y(t)$ is taken at the opposite side. Simulation is accelerated and runs for a total time of 10 seconds while a heat leakage fault is initiated at time $t = 3s$, which is modeled by

$$h(y,z,t) = -5y(t)e^{-200(z-0.5)^2}\left(1 - e^{-0.2(t-3)}\right) \qquad for \;\; t \geq 3.$$

The following observer is used to monitor the system state and output

$$\hat{x}(z,t) = x_1(z,t) + x_2(z,t) + \sum_{i=1}^{p} \hat{w}_i(t)\text{¤}_i(z,t),$$

$$\hat{y}(t) = \hat{x}(0,t),$$

with the filters defined in (13),(14), (16),(17), and (24),(25). The vector of fault parameters $\hat{w}_i$ is initialized at zero and the vector of fault basis functions is selected in the form of RBF that can span the entire space $z \in (0,1)$ as follows

$$\text{±}_i(y,z) = y(t)e^{-2p^2(z-i/p)^2} \quad for \;\; i = 1, \ldots, p-1,$$

where p is selected to be 10. The estimated fault parameters are not updated before the detection of a fault, while they will be tuned with the update law in (30) after the detection.

Figure 1 shows the solution of the distributed parameter system defined by (33), (34) with a constant input. It can be observed that the system state reaches a steady state after about 2 seconds, but when the fault is initiated at time $t = 3s$, the system state behavior changes. As expected, the leakage fault results in lower temperature at the output side.

The filter-based observer is utilized to estimate the system state and output online. The evolution of estimated state with respect to time and position is shown in Figure 2. The state estimation is initially inaccurate due to different initial conditions, but the estimation converges to the actual values in a short time. In order to perform fault detection, residual is generated by comparing the estimated and actual states at position $z = 0$.

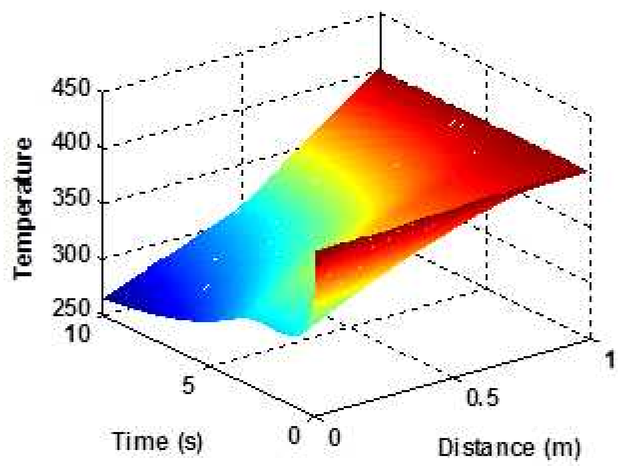


Figure 1: Actual system response

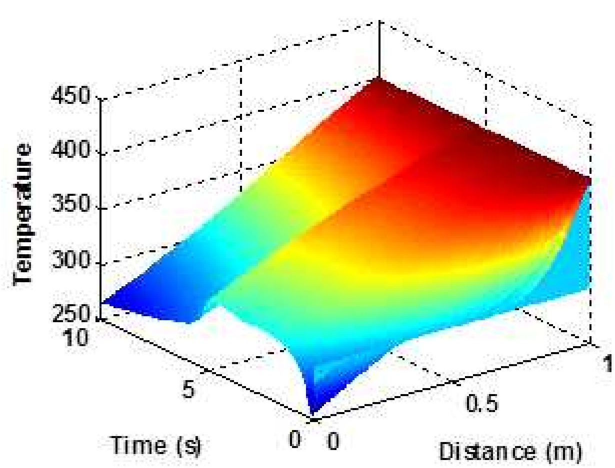


Figure 2: Estimated distributed temperature.

The residual is plotted in Figure 3 along with the detection threshold which is obtained from (22) by using $\eta = 0.2$. The residual is initially large due to incorrect initial conditions, but the stable observer derives the residual to zero under healthy operating conditions. The time-varying detection threshold prevents a false alarm in the beginning while increasing the possibility of detection when the observer error reaches its steady state value. The residual starts to increase upon initiation of fault at $t = 3s$ and the fault is declared active when the residual exceeds the threshold which happens at $t = 4.6s$. The adaptive update law for tuning the estimated fault parameters is activated immediately after detection. Subsequently, the learning algorithm captures the effect of fault on the system which causes the residual to decrease again, even in the presence of fault. The actual and estimated fault magnitudes are shown in Figure 4. The estimated fault remains at zero until the fault is detected, since the fault parameter vector is initiated at zero and the update law is only activated upon detection.

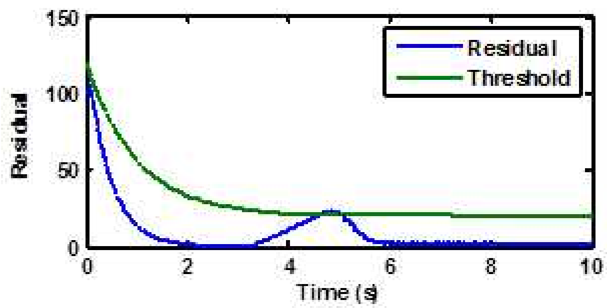


Figure 3: Detection residual and threshold

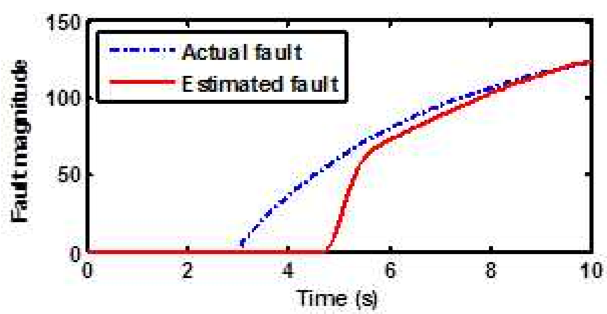


Figure 4: Actual and estimated fault magnitude

Fluid pipeline: Fluid flow and fluid pressure dynamics in a pipe are modeled by hyperbolic partial differential equations. However, assuming that the flow variation through the pipe is slow, the behavior of the system can be represented by a parabolic PDE in the following form

$$\frac{\partial p^2}{\partial t} = \frac{D^3 (ZRT)^2 \pi}{16fQ} \frac{\partial^2 p^2}{\partial z^2} + \frac{2g\pi D^3 sin\theta}{16fQ} \frac{\partial p^2}{\partial z}$$

where $p$ is the distributed variable, pressure, and $z$ is the distance from the start point of the pipe. A horizontal gas duct with the length of 100km is considered here. Table 1 provides the names of other parameters in the equation along with the values used in the simulations.

Table 1: Fluid pipeline system parameters

| Parameter | Description | Value |
|---|---|---|
| $D$ | Pipe diameter | 0.6 m |
| $f$ | Friction factor | 0.003 |
| $R$ | Gas constant | 392 $m^2/s^2K$ |
| $Z$ | Compressibility factor | 0.872 |
| $T$ | Temprature | 278 K |

The pressure applied by pump at the start of line and flow demand at the output determine the boundary conditions for the PDE equation. A pump fixes the inlet pressure at 50 bars and Figure 5 shows the output flow demand. Assume the only available measurement is the output pressure. A leakage fault is seeded 50km away from the start of pipe, 50 hours into the simulation. The fault is created by $h_{lk}(z) = Me^{-\frac{(z-l)^2}{5\times10^7}}$, where $l = 50km$ and $M$ is the magnitude of leakage fault.

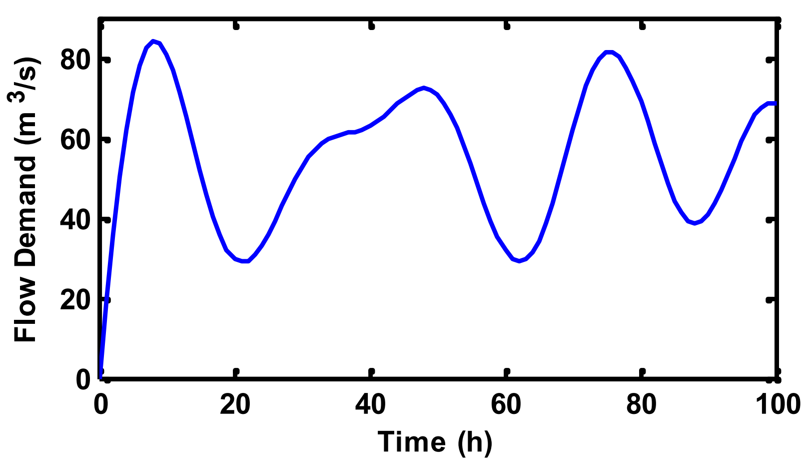


Figure 5: Flow demand at the pipe outlet

The filter-based fault detection observer is designed based on equations (13)-(20). The observer runs in parallel with the actual system and provides estimate of pressure at all points in the pipe even though the only available measurement is the pressure at the end of the pipe. Figure 6 shows the actual fluid pressure inside the pipe and Figure 7 shows the estimated fluid pressure provided by the observer with inactive fault estimator. It can be observed from the figures that the observer provides an accurate estimation of pressure before the occurrence of fault ($t < 50$), while the estimated pressure cannot follow the changes in the actual pipe in the presence of fault ($t > 50$). This difference in the estimation of state is used for fault detection. The residual which is calculated as the difference between actual and estimated output pressure is plotted in Figure 8 along with the detection threshold. The residual exceeds the threshold about 26 hours after the occurrence of fault which is the time of detection.

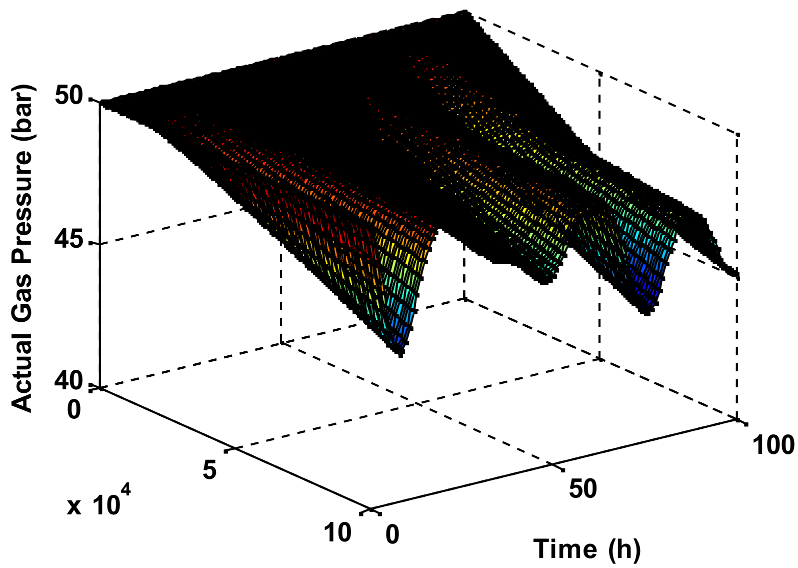


Figure 6: Actual fluid pressure along the pipe

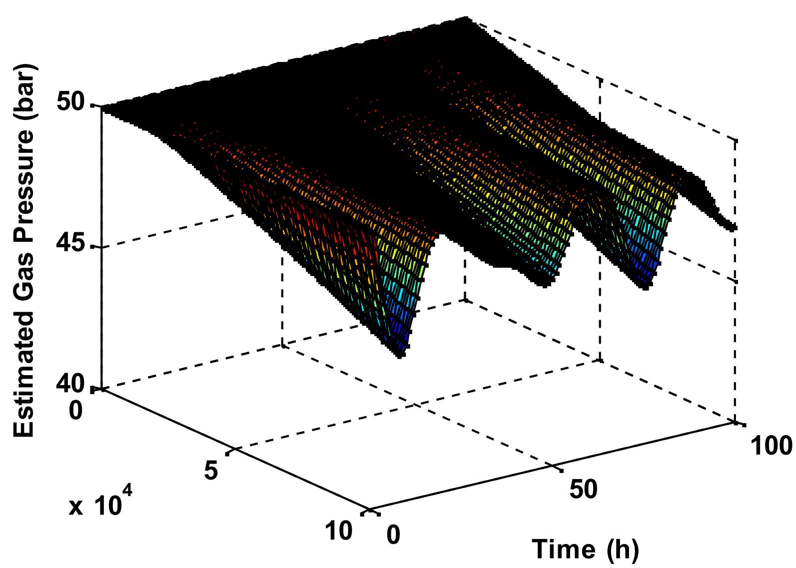


Figure 7: Estimated fluid pressure along the pipe

In this example, the magnitude of fault is assumed to be known and the estimation is performed on leakage location. This might be a strong assumption, but the proposed method cannot accurately estimate more than one fault parameter in a distributed system by only using one measurement. Figure 9 shows the actual and estimated fault location with the aforementioned assumption. The estimation is very accurate, nevertheless, future work on this topic includes investigation of fault estimation by using multiple sensors.

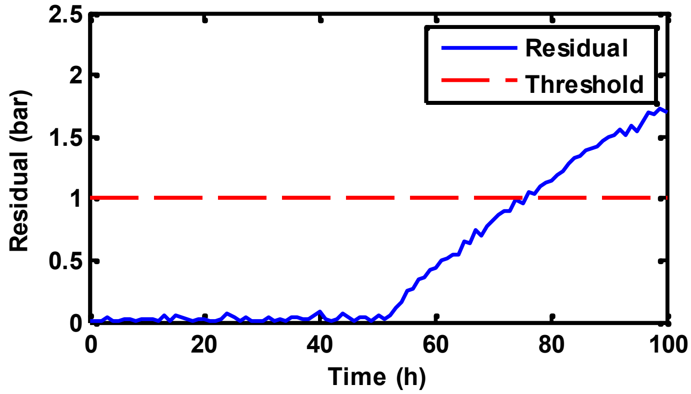


Figure 8: Detection residual and threshold

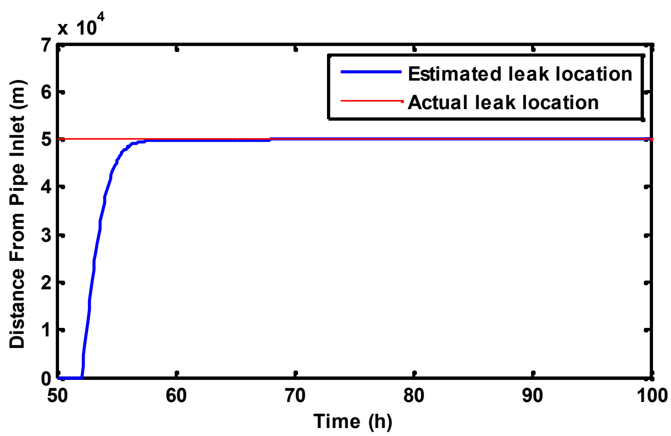


Figure 9: Actual and estimated leak location

**Conclusions**: Since the PDE model is directly used to construct the detection observer, the proposed scheme is more accurate in estimating the system states and thus highly reliable in performing fault detection than the existing fault diagnostic methods for distributed parameter systems. As seen in the simulation examples, it can also provide useful information about the unavailable system states. It was shown that if the stability conditions are satisfied with proper selection of design parameters, the observer will track the actual system states in healthy conditions and the adaptive update law will learn the unknown parameters of fault with a bounded error. Accurate and fast fault detection enhances the reliability, decreases the maintenance costs, and increases the system availability.

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